THE SIZES OF SETS ON THE REAL LINE
(TAMANHOS DE CONJUNTOS NA RETA REAL)

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December 8, 2024
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ABSTRACT

When contrasting Euclid's principle that a part is smaller than the whole with Cantor's concept of a one-to-one correspondence, where subsets can be matched with the larger set, a paradox emerges that continues to puzzle scholars to this day. To maintain Euclid's principle and challenge Cantor's one-to-one correspondence, I introduce a theoretical framework that enables the computation of the sizes of finite and infinite subsets of the Real line. This current work supports the author's prior findings.

***K*eywords** Cantor · Size · Infinite · Real · Paradox

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Conflict of Interest

The author of this manuscript affirms that there are no conflicts of interest to disclose regarding the publication of this article. The author has no financial or personal relationships that could inappropriately influence or bias the work reported in this manuscript.

1 Introduction

In this study, we introduce a method to quantify the magnitude of both finite and infinite sets of numbers on the Real line. Cantor and Hume [5] stipulated that two infinitely countable sets possess the same "size" (or cardinality) if they can be matched in a one-to-one relation. Nevertheless, this definition gives rise to a paradox when juxtaposed with Euclid's principle that a part is inherently smaller than its whole. To grasp this paradox, one need only reflect on the fact that the set of even natural numbers is indeed a proper subset of the set of all natural numbers. Nonetheless, these two sets can be matched in a one-to-one relation using a function that pairs each natural number with its double. Essentially, while Euclid would assert that the set of even natural numbers is smaller than the entire set of natural numbers, Cantor and Hume's perspective implies that both sets possess identical size or cardinality.

Initially, we will develop the intuitive understanding of the theory we propose before delving into its broader and more comprehensive version. We will additionally derive practical results for determining the sizes of sets of numbers along the Real line. We direct the reader to the author's earlier works [7, 8], where various valuable concepts are discussed in detail. Moreover, we stress that while some ideas may bear partial resemblance to those in an internet-published study, the present work was conceived independently.

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2 Theoretical Framework

In a previous publication [7], I highlight the significance of evaluating the sizes of infinite sets in Western mathematical thought. Nevertheless, the paradoxes that arise trace back to antiquity, prompting numerous esteemed figures like Galileo, Proclus, Thabit ibn Qurra, Leibniz, Emmanuel Maignan, Bolzano, Cantor, among others, to engage deeply with the topic [5].

As per [5, p. 617], [7], Galileo and Leibniz believed that it was impossible to quantify the sizes of infinite collections. Maignan, however, “defends the existence of infinite collections and the existence of different sizes among infinities” [5, 617]. Lastly, Bolzano and Cantor assert the plausibility of developing such a theory. According to [5, 626], [7], it was Dedekind who characterized infinite sets as those capable of being matched in a one-to-one correspondence with subsets of themselves. But we owe Cantor the use of one-to-one correspondences to “analyze the notion of size of infinite sets” [5, 626], [7].

In other words, for Cantor “all infinite sets of natural numbers have the same cardinality, we would say the same ‘size’”, since they can be placed in a one-to-one relation with each other [5, 627], [7]. Nonetheless, the conflict with Euclid’s principle is apparent, as Euclid believed that a part should always be smaller than the whole [5, 639]. As highlighted in [7], even natural numbers can be matched in a one-to-one relation with the entire set of natural numbers, despite being a proper subset of it. In essence, based on Cantor-Hume’s perspective, they are deemed to possess identical sizes; however, Euclid’s viewpoint suggests they should have distinct sizes.

As mentioned in my previous work [7], attempts to solve this problem exist, but it is an open question and Gödel stated that “Cantor’s theory of infinite set size is inevitable”, since “The number of objects belonging to some class does not change if, leaving these objects the same, someone changes in some way their properties or mutual relations (for example, their colors or their distribution in space)” [5, 637]. He was undoubtedly alluding to Cantor’s one-to-one relation.

3 Methodology

In the present study, I use traditional Mathematics methods and some Philosophy methods. According to [7] and [4], the methods of Philosophy are “the methods of reasoning and analysis that seek to clearly define the concepts used, investigate and expose the foundations of ideas and theories and build a systematic theory that is based on other ideas and systems of thought”.

In this context, it is essential to precisely define concepts, critique ideas, and compare them, emphasizing their limitations or strengths [4, 177]. Similarly, strive to uncover and elucidate the underlying principles of ideas, thereby laying the groundwork for a thorough critique. Essential to this study is the development of an integrated system that provides explanatory and insightful value. Hence, this study aligns with the positivist school of thought, employing “logic, reason, rigor, and inference in the pursuit of knowledge” [4, 189]. As per [7], “it also incorporates both deduction and induction, thereby drawing both ‘necessary’ and ‘probable’ conclusions [4, 190], aiming to both challenge an existing theory and formulate a new one.”

4 Results and Discussion

4.1 Intuitive Developments

4.1.1 *Size of the interval [0,1) of the Real line*

Take into account the portrayal of the infinite subdivision of the Real interval $[0, 1)$ by the balanced binary tree depicted in Figure 4.1.1.

What this binary tree signifies is an infinite division of the real interval $[0, 1)$. The root node, with a height of $h = 0$, symbolizes the complete range $[0, 1)$. We split this interval in half, generating two child nodes at height $h = 1$. The left node signifies the sub-interval $[0, 1/2)$, while the right one represents the sub-interval $[1/2, 1)$. We further split each of these sub-intervals in half. As offspring of the left node, we form the sub-intervals $[0, 1/4)$ and $[1/4, 1/2)$, while as offspring of the right node, we establish the sub-intervals $[1/2, 3/4)$ and $[3/4, 1)$. We continue this process indefinitely, continuously dividing the leaf nodes in half at each stage. This results in the formation of two child nodes for each parent node, each with half the size of the original.

It is evident that the number of sub-intervals at height h is 2^h. Additionally, the width of each of these sub-intervals is 2^{-h}. Now, some crucial verifications. Firstly. It is evident that, at every height h, the collection of sub-intervals (the

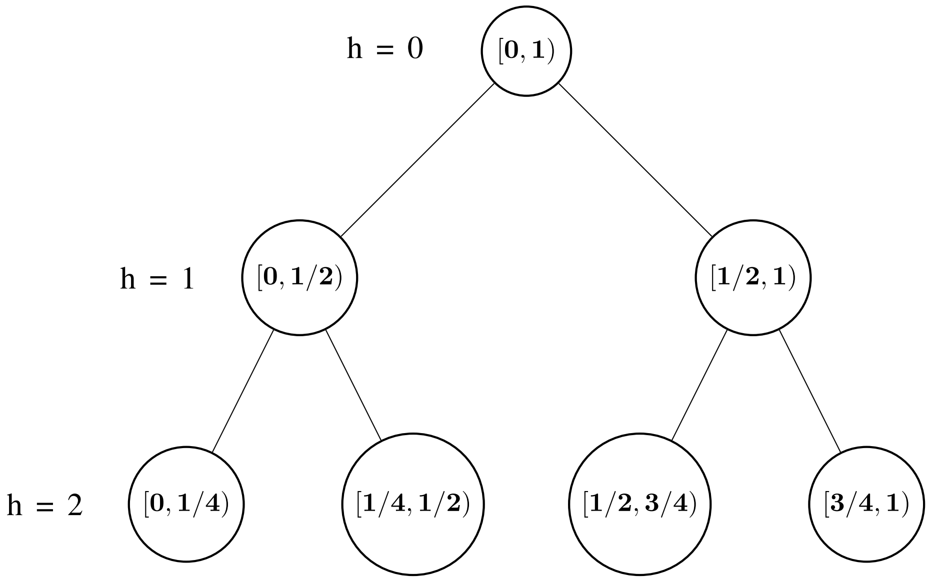

Figure 1: Balanced Binary Tree. Infinite Subdivision of the Interval [0,1)

leaf nodes of the tree at that height) spans the entire real interval $[0, 1)$. Given that there are no gaps in the union of sub-intervals, no real numbers are omitted. Likewise, it is apparent that the widths of the sub-intervals diminish towards zero (0) as we advance in building the tree, meaning, $lim_{h\to 0}(\delta = 2^{-h}) \to 0$ where δ represents the width of each sub-interval at height h. Since each sub-interval at height h invariably contains at least one real number, given that at least one of its boundaries is closed (the sole exception being the far-rightmost sub-interval with both boundaries closed), it is straightforward to deduce that, in the limit, the count of leaves in the tree corresponds to the quantity of real numbers in the interval $[0, 1)$, specifically, $QReal[0, 1) = \lim_{h\to\infty} 2^h = 2^\infty$. In simple terms, the leaves of the tree depict a division of the interval $[0, 1)$ into non-intersecting sub-intervals with a width approaching 0. Since every interval always encompasses a real number, and given that these intervals, with at least one real number each, approach zero width in the limit, they ultimately consist of single-point sets. For, having at least two real points would mean that the leaf node interval would have positive width $((\delta = 2^{-h}) > 0)$, but, in the limit, $\delta \to 0$, and this is contradictory. By ensuring no exclusion of any real numbers within the range $[0, 1)$, the quantity of leaf nodes approaching the limit consequently mirrors the tally of real numbers spanning the interval $[0, 1)$.

This formula not only signifies the potential quantity, being a constructed representation, but also the minimum number of real numbers within the interval $[0, 1)$. Initially, contemplate all real numbers within the interval $[0, 1)$. This range has a midpoint at $1/2$. When aiming to count the real numbers within $[0, 1)$, starting with a selection from the sub-interval $[0, 1/2)$, it's evident that at least one number must also be chosen from the sub-interval $[1/2, 1)$. Contemplating the midpoints of these sub-intervals, it becomes apparent that we must inevitably select at least one real number from each of them. Otherwise, gaps would be present in the real number line, and not all numbers in the initial interval $[0, 1)$ would be accounted for. As you can see, the formula $QReal[0, 1) = \lim_{h\to\infty} 2^h = 2^\infty$ further forms the lower threshold of the count of real numbers in the interval $[0, 1)$. This is evident as, during the subdivision of sub-intervals, we indispensably choose at least one real number from each offspring of these sub-intervals; otherwise, gaps would emerge on the real number line within the interval $[0, 1)$. Put simply, to enumerate the total real numbers within the interval $[0, 1)$, the count of real numbers that we must inevitably choose expands, at the very least, exponentially following the formula 2^h, where h stands for the quantity of midpoints $(1/2, 1/4, 3/4, etc.)$ employed to partition the sub-intervals within the interval $[0, 1)$ while tallying the total number of real numbers encompassed within it. Initially, we select the midpoint $1/2$ and establish two sub-ranges. Subsequently, we further subdivide each of them at $1/4$ and $3/4$, generating 4 sub-intervals within the interval $[0, 1)$, and so forth. Essentially, the binary tree employed (Figure 4.1.1) is not only adequate for enumerating the total real numbers within the interval $[0, 1)$ but also indispensable. It signifies the minimal tally and naturally emerges during the enumeration of real numbers within the interval $[0, 1)$.

Another aspect to verify initially is that all the boundaries of the sub-intervals denoted by the tree nodes are rational numbers. Let's explore how we can depict each of these boundaries, in other words, each of these rational numbers, utilizing a binary representation of fractional values. Consider, first, the height $h = 0$. At this stage, the tree solely consists of the root node with boundaries at 0 and 1. These boundaries, these rational numbers, are depicted by the non-fractional binary numbers $(0.)$ and $(1.)$. Let's proceed to height $h = 1$. At this point, we have two sub-intervals

whose endpoints are 0, and 1. These boundaries, or rational numbers, can be expressed in binary as 0.0, 0.1, and 1.0, signifying $0 + 0 * (1/2)^{-1} = 0$, $0 + 1 * (1/2)^{-1} = 1/2$ and $1 + 0 * (1/2)^{-1} = 1$. Let us now consider height $h = 2$. At this point, we have four sub-intervals with endpoints at 0, $1/4$, $1/2$, $3/4$, and 1. These rational numbers can be represented by the binary numbers 0.00, 0.01, 0.10, 0.11 and 1.00, that is, $0 + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} = 0$, $0 + 0 * (1/2)^{-1} + 1 * (1/2)^{-2} = 1/4$, $0 + 1 * (1/2)^{-1} + 0 * (1/2)^{-2} = 1/2$, $0 + 1 * (1/2)^{-1} + 1 * (1/2)^{-2} = 3/4$, $1 + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} = 1$.

In general, at height h, we will encounter 2^h sub-intervals with $2^h + 1$ rational boundaries, which can be portrayed through all fractional binary numbers composed of h binary digits after the binary point, alongside the inclusion of the number 1. An essential inference drawn from this binary depiction of the rational boundaries of the tree's leaf nodes is that, for every height h of the tree, there exists a one-to-one correspondence between the rational boundaries of the sub-intervals and each of the fractional binary representations comprising h binary places, along with the inclusion of the number 1. In other words, the rational boundaries of the leaf nodes at height h are all rational numbers with a maximum of h binary digits following the binary point. As we progress with the construction of the tree, augmenting its depth, $h \to \infty$, meaning that the quantity of binary places approaches infinity. Consequently, the rational numbers denoted by these binary notations, serving as the boundaries of the sub-intervals, encompass, in the limit, all the rational numbers within the interval $[0, 1]$ (including the number 1). Given that the quantity of boundaries of the sub-intervals in the tree at height h is $2^h + 1$, signifying all rational numbers with h decimal digits in binary form, the overall count of rational numbers within the interval $[0, 1]$ is $QRac[0, 1] = \lim_{h\to\infty}(2^h + 1) = 2^\infty$.

Now, let's observe how straightforward it is to derive a listing of all real numbers in the interval $[0, 1]$ through enumerating the nodes of the binary tree. The enumeration of the tree's nodes follows a simple process: beginning with assigning the number 0 to the root node, then allocating numbers 1 to 2 to the root node's children, and subsequently assigning numbers 3 to 6 to the nodes at level 2, and so on, in a zigzag descending manner. This method enables us to create a comprehensive enumeration of all nodes in the binary tree, covering its leaf nodes as well. As the leaves (nodes at level h) of the binary tree, ultimately representing all the real numbers in the interval (each indicating a single number), we establish an enumeration of all the real numbers within the interval $[0, 1]$. This appears to challenge Cantor's assertion in his theory that establishing a one-to-one relation between real numbers and natural numbers is unattainable.

4.1.2 *Size of unlimited intervals of the Real line*

We will utilize the interval $[0, 1)$ as a unit (similar to a standard unit of measurement in Physics) and build infinite intervals with this base unit. Consider, first, the interval $[0, n)$. Let us divide this interval into sub-intervals $[0, 1), [1, 2), \ldots, [n - 1, n)$, up to any natural number $n > 0$. Every one of these sub-intervals can be illustrated by a binary tree, as depicted, of height h. Hence, the count of numbers (nodes at height h, including the number 1) denoted by these intervals is $QReal[0, n] = 2^h * n$. Therefore, the count of $QReal$ numbers on the ray $[0, \infty)$ is, transiting to the limit, $QReal[0, \infty) = \lim_{h\to\infty, n\to\infty}(2^h) * n = \infty * 2^\infty$.

Similarly, the number of real numbers in the negative real ray is $QReal[0, -\infty) = \lim_{h\to\infty, n\to\infty}(2^h) * n = \infty * 2^\infty$. Likewise, if we follow a comparable approach involving both rays, we deduce that the quantity of real numbers on the entire Real line is:

$$QReal(-\infty, +\infty) = QReal[0, +\infty) + QReal[0, -\infty) - 1 =$$
$$\lim_{h\to\infty, n\to\infty}(2^h) * n + \lim_{h\to\infty, n\to\infty}(2^h) * n - 1 =$$
$$2 * \lim_{h\to\infty, n\to\infty}(2^h) * n - 1 = \lim_{h\to\infty, n\to\infty}((2^{h+1}) * n) - 1 =$$
$$2 * \infty * 2^\infty$$

Let's briefly note that, if we have an interval $[a, b)$ on the Real line, we can express the count of real numbers within it as $QReal[a, b) = QReal[0, 1) * (b - a)$. In this manner, we can ensure the desired property that "the quantity of numbers within a set of real numbers is the sum of the quantities of each subset of a partition of that set."

4.2 Theoretical Developments

According to [1] and [9], we can define that, given a subset U of the Real line, the diameter of U can be defined as $D(U) = \sup \mid (x - y) \mid$, x, y belonging to U. In other words, it is the greatest distance between any points of U. If U_i is an enumerable collection of sets of maximum diameter δ that cover F, that is, F is contained in the $\bigcup_i^\infty U_i$, with $0 < D(U_i) \le \delta$ for each i, then $\{U_i\}$ is a $\delta - coverage$ of F. In our case, we will use as sets the segments of the Real line closed on the left and open on the right $[a, b)$, so that adjacent segments do not intersect.

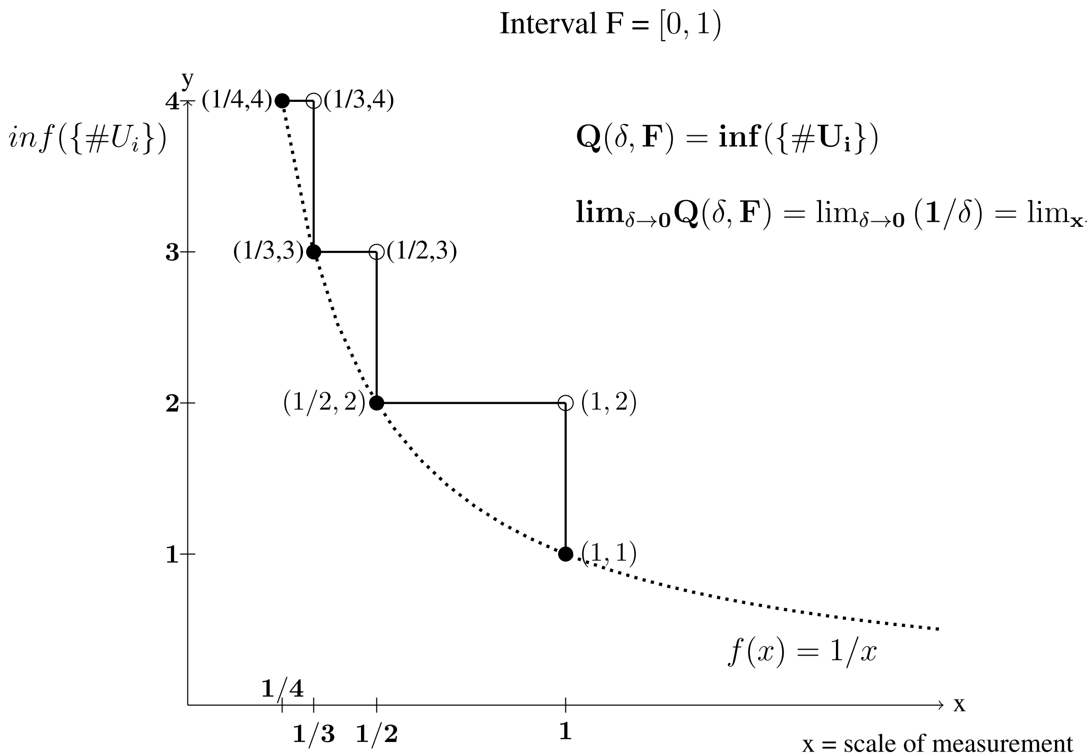

Figure 2: Quantity of Reals in $[0, 1)$

Suppose that F is a subset of points on the real line $\mathbb{R}$. We define $Q(\delta, F) = \inf \#\{U_i\}$, where $\#U_i$ is the number of elements in the set U_i. Although not necessary, we still assume that the intersection of U_i with U_j is empty, for $i \neq j$. This doesn't change $\#\{U_i\}$ since, if any two subsets U_i intersect, it is possible to reduce one of them by deleting the intersection, in order to maintain the $\delta - coverage$ and do not increase the value $\#\{U_i\}$. Finally, we can define the number of real numbers in the set F as $QReal(F) = \lim_{\delta\to 0} Q(\delta, F)$.

Let's start with the calculation of $QReal([0, 1)) = \lim_{\delta\to 0} QReal(\delta, [0, 1))$. It is easy to see that for the interval $[0, 1)$, starting with $\delta = 1$ and decreasing it, we obtain the ladder-shaped graph, below (Figure 2), for the value of $Q(\delta, [0, 1)) = \inf \#(\{U_i\})$. That is:

$$QReal([0, 1)) = \lim_{\delta\to 0} Q(delta, [0, 1)) = \lim_{\delta\to 0}(1/\delta) =$$

$$\lim_{x\to 0}(1/x) = \infty$$

According to this definition, it is also easy to see that if $A = B \cup C$, and $B \cap C = \emptyset$, $QReal(A \cup B) = QReal(A) + QReal(B)$. Consider $B = [a, b)$ and $C = [c, d)$, such that $(c - b) \geq k$. For all $\delta < k$, any $\delta - coverage$ of the $B \cup C$, U_i, it will be such that none of the U_i will intersect both, B and C. That is, $\#\{U_i \cap B\} + \#\{U_i \cap C\} = \#\{U_i \cap (B \cup C)\}$, such that $\inf(\#\{U_i \cap B\}) + \inf(\#\{U_i\} \cap C) = \inf(\#\{U_i \cap (B \cup C)\})$, and, therefore,

$$QReal(A = (B \cup C)) = \lim_{\delta\to 0} Q(\delta, B \cup C) =$$

$$\lim_{\delta\to 0}(\inf(\#\{\{U_i\} \cap (B \cup C)\})) =$$

$$\lim_{\delta\to 0}(\inf(\#\{\{U_i\} \cap B\}))+$$

$$\lim_{\delta\to 0}(\inf(\#\{\{U_i\} \cap C\})) =$$

$$QReal(B) + QReal(C)$$

As we are interested in $\delta - coverage$ such that $\delta \to 0$, for any $B = [a, b)$ and $C = [c, d)$, such that $c - b \geq k > 0$, we can make δ as small as necessary for the present theorem to become true. And, doing $k \to 0$, we conclude that

the theorem is valid whenever $B \cap C = \emptyset$. Another more generic way of looking at this is to realize that, given a $\delta - coverage$ $\{U_i\}$ of $B \cup C$, we can build the $\delta - coverage$ $\{Wi\} = (\{U_i\} \cap B) \cup (\{U_i\} \cap C)$ in which larger segments of the original coverage will be sectioned into smaller segments (in this case, only one original segment, which intersects both B and C, will be sectioned into two smaller segments). This is not a problem for calculating the limit when $\delta \to 0$, because, although we have increased the number of segments of the original coverage (in this case, just one additional segment), when we want to calculate the minimum, the larger original segments (those that intersect both B and C) would be eliminated, and with them, the original coverage, for small enough delta. That is, given any coverage $\{U_i\}$, we get coverage $\{Wi\}$, which does not change, when we go to the limit, the calculation, so that:

$$QReal(A = (B \cup C)) = \lim_{\delta \to 0} Q(\delta, B \cup C) =$$

$$\lim_{\delta \to 0}(\inf(\#\{\{Wi\} \cap (B \cup C)\})) =$$

$$\lim_{\delta \to 0}(inf(\#\{\{Wi\} \cap B\})) + \lim_{\delta \to 0}(inf(\#\{\{Wi\} \cap C\})) =$$

$$QReal(B) + QReal(C).$$

Based on this result, it is easy to see that, for any subsets, A, B and C of the Real line, such that $A = (B \cup C)$, $QReal(A) = \{QReal(B) + QReal(C)\} \setminus QReal(B \cap C)$, as $(B \cup C) = ((B \setminus C) \cup (C \setminus B) \cup (B \cap C))$. As each of these three sets is mutually disjoint with the other two, we can apply the theorem derived here and obtain the desired formula for the union of any sets on the Real line.

Note that the theory developed here allows calculating the size of any subset of real numbers. Let's perform the calculation for the set of all real numbers, that is, for the continuum, the entire Real line. We have already seen that

$$QReal([0,1)) = \lim_{\delta \to 0} Q(\delta, [0,1)) = \lim_{\delta \to 0}(1/\delta) =$$

$$\lim_{x \to 0}(1/x) = \infty.$$

In turn, we will assume that the distribution of real numbers on the Real line is homogeneous, which is obvious, since this is a basic property of the idea of quantity (especially unit quantity), which is an inherent and inseparable property in the notion of number, whatever its type. In other words, we are assuming that the density, calculated by the division of the quantity of numbers to the difference of boundary values is constant. Hence, given any real interval $[a, b)$, $QReal([a, b))/(b - a) = QReal([0, 1))/(1 - 0)$, which implies that $QReal([a, b)) = \lim_{x \to 0}((b - a)/x))$.

In addition to preserving the theorem for calculating the size of the union of sets, derived above, this formula allows the calculation of the size of the entire Real line. See that $QReal([0, \infty)) = \lim_{b \to \infty, x \to 0}(b - 0)/x = \lim_{y \to \infty, x \to 0}(y/x)$. Similarly, using the union theorem, $QReal([0, -\infty)) = \lim_{y \to \infty, x \to 0}(y/x)$, and $QReal((-\infty, +\infty)) = 2 * (\lim_{y \to \infty, x \to 0}(y/x)) - 1$. The (-1) is to prevent the number 0 from being counted twice in the positive and negative rays. We can also calculate the size of the first quadrant of the plane $\mathbb{R}^2$, defining $QReal(A \times B) = QReal(A) * QReal(B)$. This can be easily demonstrated since any $\delta - coverage$ is a countable set. If we perform the Cartesian product of two of these sets we obtain another $\delta - coverage$, this time in the plane, by a set whose size is the product of the sizes of the original $\delta - coverages$. Then we take the smallest of these two sides of the equation and let them to the limit when $\delta \to 0$ and we obtain the desired formula. Thus, $QReal([0, \infty) \times [0, \infty)) = \lim_{y \to \infty, x \to 0}(y/x)^2$. And, for the Real plane, $QReal((-\infty, \infty) \times (-\infty, \infty)) = (4 * \lim_{y \to \infty, x \to 0}(y/x)^2) - 4 * \lim_{y \to \infty, x \to 0}(y/x) + 1$. The $(-4 * \lim_{y \to \infty, x \to 0}(y/x))$ is to avoid double counting the $[0, \infty)$ rays, and the $+1$ it to account for the multiple inclusions and exclusions of the point $(0, 0)$ in the previous operations.

To calculate the size of a subset of integers $\mathbb{Z}$ according to the present theory is not difficult. Just realize that for any $\delta - coverage$ $\{U_i\}$ of a set of integers $\mathbb{Z}$ in the Real line, $\lim_{\delta \to 0} \inf \#(\{U_i\})$ = **"Number of integers in the set"** $= \#\mathbb{Z}$. This result corroborates the author's formula for calculating the relative size of infinite sets of natural numbers using asymptotic density [7].

Let us now look at the case of rational numbers. Rational numbers are a subset of real numbers, such that $(QReal(\mathbb{Q}) = QRac(\mathbb{Q})) \leq QReal(\mathbb{R})$. Furthermore, it is easy to verify that, for the rational numbers of $[0, 1)$, the function shown in Figure 2, already explained, follows. So that $QRac([0, 1)) = \lim_{\delta \to 0} Q(\delta, [0, 1)) = \lim_{\delta \to 0}(1/\delta) = \lim_{x \to 0}(1/x) = \infty$. With this result, the entire present theory, as it applies to any subset of real numbers, also applies to rational numbers. Shouldn't rational numbers be a proper subset of real numbers, that is, irrational numbers don't exist? If this theory is correct, we will analyze this issue in a future study.

As for the formula $QReal[0, 1) = \lim_{h \to \infty} 2^h = 2^\infty$, developed in the initial part of the study, it is just a special case of the formula $QReal([0, 1)) = \lim_{\delta \to 0} Q(delta, [0, 1)) = \lim_{\delta \to 0}(1/delta) = \lim_{x \to 0}(1/x)$, when we do $x = 2^{-n}$ and $n \to \infty$. As they are also when we do $x = 3^{-n}, 4^{-n}, 5^{-n}$ etc. All of these formulas conform to the generic formula that emerges from the graph in Figure 2.

It is important to realize that all these quantities are computed taking the limits of formulas based on a given variable δ, which is the maximum diameter of the sets in a given coverage of the set to be counted. As this δ is made to approach 0, it can be interpreted as a number that indicates the accuracy of the "measurement" of the quantity of real numbers in the set at a given scale ($\delta - coverage$). Therefore, we can interpret the formula for $QReal([0,1)) = \lim_{x \to 0}(1/x)$ as counting the real numbers in $[0,1)$ with increasing accuracy, as the scale of "measurement" (x) goes from $1 \to 0$.

5 Final Considerations

In this study, we introduce a methodology for gauging the sizes of subsets, whether finite or infinite, of the Real line that does not rely on Cantor's one-to-one relation. On the contrary, the results seem to contradict the essence of this one-to-one relation, as previously argued in [7].

The research offers an intuitive perspective on the topic, while also unveiling a meticulous theoretical framework. The theory outlined enables us to derive formulas whose limits correspond to the magnitudes of subsets of Real, Rational, Integer, or Natural numbers. Possibly, the formulas hold more significance than the limits themselves, as they enable comparisons of the sizes of sets even if infinite. Ultimately, intuition of Euclid's principle, that the size of the proper part must be smaller than the size of the whole, is preserved.

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